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Transfer behavior of a class of generalized N-diffusion equations in a semi-infinite medium ☆

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Abstract

Suitable similarity transformations were used for reducing the generalized N-diffusion equations to a class of singular nonlinear boundary value problems. Similarity solutions are presented analytically and numerically. The results indicated that for each fixed α, the general diffusion flux $\theta(s)$ decreases with the increase of the power law N and sharply with the increase of σ. For $0 < \sigma \leqslant 1$, $\theta(s)$ decrease with the increase of α, however, the behavior is quite opposite for $\sigma > 1$.
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1. Introduction

Transfer problems are of great interest in a wide range of natural problems and industrial applications. In the sense that some entity is transferred under the gradient of a concentration-like quantity, we have, for the processes of this nature, the general relation $\boldsymbol{Q} = A \cdot \boldsymbol{B}$. Here $\boldsymbol{Q}$ is the flux density vector, $\boldsymbol{B}$ is the vector function of concentration-like gradients, and A is a coefficient in the function of time, position and/or concentration.

In this paper we consider a generalized form of diffusion for which $\boldsymbol{B} = |\nabla u|^{N-1}\nabla u$ and $A = k(u)$. This study serves as an introduction to further works deal with the concentration-dependent N-diffusion and certain forms of space-dependent N-diffusion. These more complicated types of N-diffusion are immediately relevant to physical problems of interest, including the unsteady vertical heat transfer from a horizontal surface by (turbulent) free convection, and the unsteady turbulent flow of a liquid with a free surface over a plane [1].

The one-dimensional (1-D) form of the N-diffusion equation is

$$\left[k(u)|u_x|^{N-1}u_x\right]_x = u_t \quad \text{in } G \tag{1}$$

$$u(0,t) = \sigma, \quad 0 < t \leqslant T, \qquad u(x,0) = 0, \quad x > 0, \tag{2}$$

where $G = \{(x,t)\colon\ x > 0,\ 0 < t \leqslant T\}$, $k(u) \geqslant 0$ in G. We limit ourselves into the physical meaning of $N > 0$. The problem can be thought of the unsteady heat conduction in a semi-infinite medium $x > 0$, initially at zero temperature. When time $t > 0$, temperature σ is applied and maintained at the extremity $x = 0$, where u denotes temperature, and $D(u) = -k(u)|u_x|^{N-1}u_x$ is the heat density per unit area.

Recently, Wang [2] and Zheng [4] have considered some generalized diffusion equations similar to (1) at certain initial and boundary conditions. Existence and uniqueness results for a positive solution are analytically established by employing the similarity transformation and perturbation technique. However, the behavior and mechanism for diffusion and transfer are not understood at present time. It may be seen that, when $N \neq 1$, the generalized diffusion equation (1) is super-nonlinear because of the term $k(u)|u_x|^{N-1}u_x$ ($N > 0$, $N \neq 1$). Therefore, the problem is difficult to study analytically and numerically. In present work, the purpose is to study the transfer behavior of Eqs. (1), (2), and the special emphasis is given to the formu-

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lation of the generalized N-diffusion equations, which provide similarity solutions.

2. Converting into two-point boundary value problem

2.1. Similarity transformation

Since the restriction of the classical solution is both irksome and unnatural in many instances, we shall deal with the solutions that may not have a derivative everywhere. It is convenient to present our problem in a so-called weak formulation.

We call a function $u(x,t)$ is a solution of Eqs. (1), (2), if $u(x,t)$ is continuous in $G=\{(x,t)\colon x>0,\ 0<t\leqslant T\}$, with $\lim_{x\to 0} u=\sigma$ and $\lim_{t\to 0} u=0$ a.e. (almost everywhere) $x>0$, and $u(x,t)$ admits a weak first derivatives which are locally integrable in G. Function $f(u)=k(u)|u_x|^{N-1}u_x$ is locally integrable in G. And for all sufficiently 'good' function $\psi(x,t)\in C_0^1(G)$ (where $C_0^1(G)$ denote the class of first continuously differentiable function that vanish on the exterior of a bounded set) such that

$$\iint_G \left\{\psi_t u-\psi_x f(u)\right\}\mathrm{d}x\,\mathrm{d}t=0 \tag{3}$$

Setting $u(x,t)=w(\eta)$ and $\psi(x,t)=g(t)\phi(\eta)$, $\eta=Cx^{\gamma}t^{\beta}$. Where C, γ, and β are constants to be determined. Substituting the above expressions into (3), by setting $C=\gamma=1$, and $\beta=-1/(N+1)$ yields

$$\int g(t)t^{-(1+\beta)}\,\mathrm{d}t\int \phi'(\eta)\left\{\beta\eta w(\eta)-f\big(w(\eta)\big)\,\mathrm{d}\eta\right\} + \big(g'(t)\cdot t^{-\beta}\,\mathrm{d}t\big)\int \phi(\eta)w(\eta)\,\mathrm{d}\eta=0 \tag{4}$$

Integrating $\int g'(t)\cdot t^{-\beta}\,\mathrm{d}t$ by parts

$$\int \phi'(\eta)\left\{-\beta\eta w(\eta)+f\big(w(\eta)\big)\,\mathrm{d}\eta\right\} - \int \beta\phi(\eta)w(\eta)\,\mathrm{d}\eta=0 \quad \forall\phi\in C_0^1(0,\infty) \tag{5}$$

Putting

$$-\int \phi(\eta)w(\eta)\,\mathrm{d}\eta=\int \phi'(\eta)\int_0^{\eta} w(s)\,\mathrm{d}s\,\mathrm{d}\eta \tag{6}$$

we obtain

$$\int \phi'(\eta)\left[-\beta\eta w(\eta)+f\big(w(\eta)\big)+\beta\int_0^{\eta} w(s)\,\mathrm{d}s\right]\mathrm{d}\eta=0 \quad \forall\phi\in C_0^1(0,\infty) \tag{7}$$

Integrating Eq. (7) by parts again and noting $\forall\phi\in C_0^1(0,\infty)$ yields

$$\int \left[-\beta\eta w(\eta)+f\big(w(\eta)\big)+\beta\int_0^{\eta} w(s)\,\mathrm{d}s\right]'\phi(\eta)\,\mathrm{d}\eta=0 \quad \forall\phi\in C_0^1(0,\infty) \tag{8}$$

Eq. (8) implies

$$\left[-\beta\eta\big(w(\eta)\big)+f\big(w(\eta)\big)+\beta\int_0^{\eta} w(s)\,\mathrm{d}s\right]'\equiv 0 \tag{9}$$

It is now easy to see that

$$f\big(w(\eta)\big)-\beta\eta w(\eta) = -\beta\int_0^{\eta} w(s)\,\mathrm{d}s+D \quad \text{a.e. in } \eta>0 \tag{10}$$

Upon a possible redefinition on a set of measure zero, (i) $f[w(\eta)]-\beta\eta w(\eta)$ is absolutely continuous locally in $\eta>0$, and (ii) $(f[w(\eta)]-\beta\eta w(\eta))'=-\beta w(\eta)$ holds almost everywhere in $\eta>0$.

As $w(\eta)$ is an absolutely continuous locally in $\eta>0$, by hypothesis on $u(x,t)$, it follows from (i), equation (ii) is equivalent to

$$\big(f\big(w(\eta)\big)\big)'=\beta\eta w'(\eta) \quad \text{a.e. in } \eta>0$$

i.e., the nonlinear boundary value problem is derived as

$$\left[(N+1)k\big(w(\eta)\big)\big|w'(\eta)\big|^{N-1}w'(\eta)\right]'=-\eta\cdot w'(\eta) \quad (0<\eta<+\infty) \tag{11}$$

$$w(0)=\sigma, \qquad w(+\infty)=0 \tag{12}$$

Conversely, if $w(\eta)$ is a solution of (11), (12), then it follows from hypotheses (i) and (ii), and the function $u(x;t)=w(\eta)$ must be the solution of generalized diffusion problems (1), (2). Therefore, in what follows, we shall pay our attention mainly on the nonlinear boundary value problems (11), (12).

2.2. Inverse function formulation

Let $s=w(\eta)$ be a solution to the nonlinear boundary value problems (11), (12). If $w(\eta)$ is strictly decreasing in $[0,+\infty)$, then $w'(+\infty)=0$, and the function $\eta=y(s)$, inverse to $s=w(\eta)$, exists. We have also $s=w(y(s))$ on $(0,\sigma)$ and $\eta=y(w(\eta))$ on $(0,+\infty)$. And $w'(y(s))=1/y'(s)$ holds in $(0,\sigma)$. Substituting $\eta=y(s)$ into Eqs. (11), (12), noting that $|w'(\eta)|^{N-1}w'(\eta)=-(-w'(\eta))^N$ and setting $\theta(s)=k(s)/(-y'(s))^N$ (where $\theta(s)>0$, $0<s<\sigma$), then we formally arrived the following singular nonlinear two-point boundary value problems:

$$\theta''(s)=-\frac{1}{N+1}\left(\frac{k(s)}{\theta(s)}\right)^{1/N} \quad (0<s<\sigma) \tag{13}$$

$$\theta(0)=0, \qquad \theta'(\sigma)=0 \tag{14}$$

where $\theta(s)=-k(s)|w'(y(s))|^{N-1}w'(y(s))$ denotes the general diffusion flux.

A general form of (13), (14) has been studied in [3,5–9] in the form of

$$\begin{cases} z''(x) = -f\big(x, z(x)\big), & 0 < x < 1 \\ \alpha z(0) - \beta z'(0) = 0, & \gamma z(1) + \delta z'(1) = 0 \end{cases}$$

where $f(x,z)$ is continuous and positive in $(0,1)\times(0,+\infty)$, and $\lim_{z\to 0} f(x,z) = +\infty$. It has been shown that this problem under appropriate conditions on $f(x,z)$ has a unique positive solution.

3. Solutions and discussions

Let $\theta(s)$ be a solution of Eqs. (13), (14) on $[0,\sigma]$. Then, it may be proved that the problem has a unique positive solution, which can be represented by the formula

$$\theta(s) = -\frac{1}{N+1}\int_0^s (s-t)\left[\frac{k(t)}{\theta(t)}\right]^{1/N} \mathrm{d}t + \frac{s}{N+1}\int_0^\sigma \left[\frac{k(t)}{\theta(t)}\right]^{1/N} \mathrm{d}t \tag{15}$$

where $\theta'(s)$ is positive and strictly decreasing in $(0,\sigma)$. Conversely, it is easy to check that a positive and absolutely continuous solution for the integral equation (15) is the solution of (8), (9). Utilizing the unique positive solution of two-point boundary value problems (13), (14), we may construct the solution $w(\eta)$ of Eqs. (11), (12), and therefore we obtained the solution $u(x,t) = w(\eta)$ of diffusion equations (1), (2).

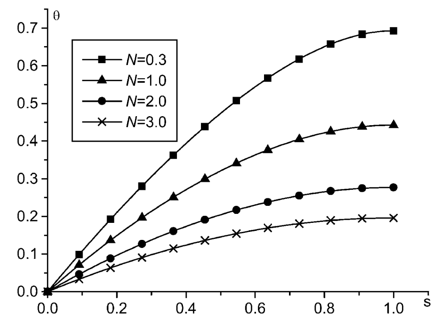

Fig. 1. Temperature distribution for $k(s) = s$, $N = 0.3$–3.0.

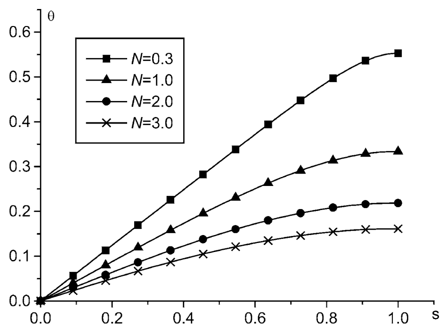

Fig. 2. Temperature distribution for $k(s) = s^3$, $N = 0.3$–3.

Eqs. (13), (14) may be solved numerically for its special cases of $k(s) = s^\alpha$, and parameters N and σ by employing the shooting technique. The numerical results are presented in Figs. 1–8. It may be seen that for each fixed $\alpha > 0$, $N > 0$ and σ, the problem has exactly one positive solution.

Figs. 1–5 show the solutions for $\sigma = 1$, $N = 0.3$–3 and $\alpha = 1$–5. It may be seen that, for each fixed α, the thermal diffusion flux $\theta(s)$ is a decreasing function of N, which means that the profiles exhibited by a big power law N possess a smaller diffusion. For each fixed parameter N, the thermal diffusion flux $\theta(s)$ decreases with an increase of α. It means that on the interval of $[0,1]$, a big α will bring a small thermal diffusion. Noted that $s^{\alpha_1} > s^{\alpha_2}$ for all $0 < \alpha_1 < \alpha_2$ and $s \in (0,1)$, which shows that the diffusion flux $\theta(s)$ increases with the increase of $k(s)$.

Figs. 6–8 reveal the character of $\theta(s)$ for $\sigma = 0.3$–1.5, $N = 3$, and $\alpha = 1$–5. It may be seen that the boundary temperature σ has a very important effect on the temperature

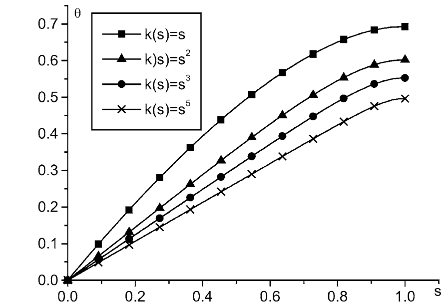

Fig. 3. Temperature distribution for $N = 0.3$, $k(s) = s^1$ to s^5.

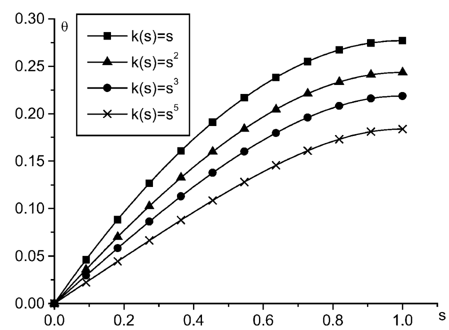

Fig. 4. Temperature distribution for $N = 2$, $k(s) = s^1$ to s^5.

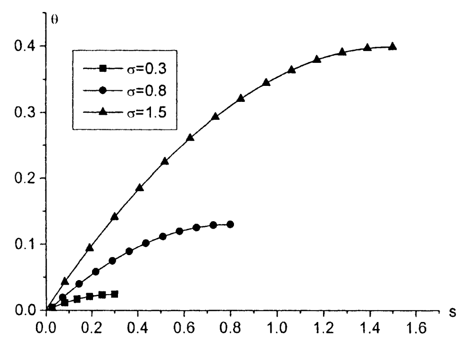

Fig. 5. Temperature distribution for $k(s) = s^5$, $N = 3$, $\sigma = 0.3$–1.5.

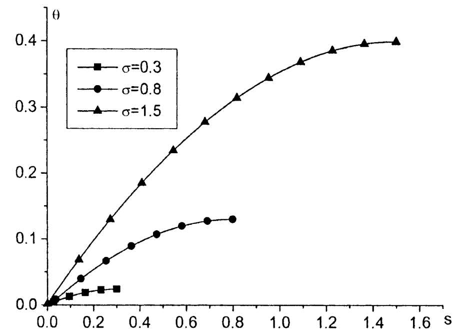

Fig. 6. Temperature distribution for $k(s) = s$, $N = 3$, $\sigma = 0.3$–1.5.

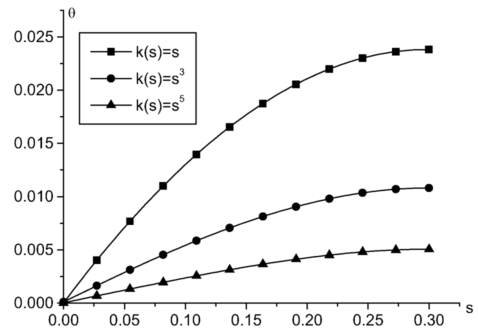

Fig. 7. Temperature distribution for $\sigma = 0.3$, $N = 3$, $k(s) = s^1$ to s^5.

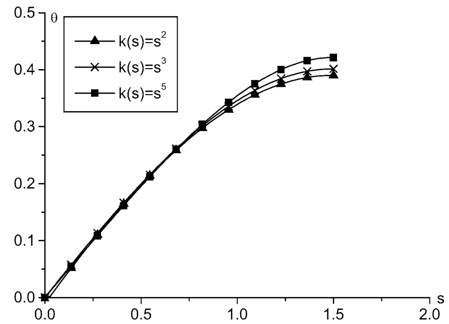

Fig. 8. Temperature distribution for $\sigma = 1.5$, $N = 3$, $k(s) = s^2$ to s^5.

distribution. The energy function $\theta(s)$ increases sharply with the increasing in σ. For $0 < \sigma \leqslant 1$, $\theta(s)$ decreases with the increase of α. However, the behavior is quite opposite for $\sigma > 1$.

4. Conclusions

Suitable similarity transformations were used to reduce the generalized N-diffusion equations to a class of singular nonlinear boundary value problems. Similarity solutions are analytically and numerically presented.

For each fixed α, the heat diffusion flux $\theta(s)$ decreases with the increase of N, i.e., a big power law N will bring a smaller diffusion. For each fixed parameter N, on the interval of $[0, \sigma]$, the heat diffusion flux $\theta(s)$ decreases with increase of α, i.e., the heat diffusion increases with the increase in $k(s)$.

The boundary temperature σ has a very important effect on the heat distribution. The energy function $\theta(s)$ increases sharply with the increasing in σ. For $0 < \sigma < 1$, $\theta(s)$ decreases with the increase of α. However, the behavior is quite opposite for $\sigma > 1$.

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